

The Royal Society of Edinburgh

Presidential Address

Mind, Matter and Mathematics

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Introduction

A Presidential address should be one of general interest to a wide audience, and in the spirit of the Scottish Enlightenment which gave birth to the Royal Society of Edinburgh.

The topic of my lecture today has always been central to philosophy, but my contribution is to include mathematics in the title. There are good reasons for this, both historical and philosophical. As a philosopher I am an amateur and there are many in this room who will be much more expert on the subject than me, but I am a mathematician and here I speak from a life-time of experience.

In early centuries many philosophers were interested in mathematics. Notable among them were Plato, Descartes, Leibniz, Kant and Bertrand Russell. In fact, until quite recent times, natural philosophy, as contrasted with moral philosophy, was often synonymous with applied mathematics. When I was a student in Cambridge almost fifty years ago our examination papers came in two sets: one labelled 'Pure Mathematics' and the other labelled 'Natural Philosophy'.

So I am treading very familiar territory, where the basic questions are:

1. What is physical reality?
2. Is knowledge innate or derived from experience?
3. What is mathematics?
4. What is the relation between mathematics and physics?
5. Where does the human mind fit in to all this?

Of course, as with all deep philosophical questions, there are no permanent and final answers. But we learn by asking questions. We can also review our understanding in the light of progress in natural science (physics, mathematics, evolution, psychology, neurophysiology ...).

I will address these questions in turn.

1. What is physical reality?

The human understanding of the physical world proceeds through various stages. First there is human perception, where we receive stimuli from the senses providing mental pictures and then our brain interprets these as objects with mutual interactions. This is a much more complex operation than it seems, as modern science has shown. Vision is the sense which has been most thoroughly explored and we now realise that, literally, there is much more to seeing than meets the eye. The raw data has to be given structure and meaning. The brain has to guess what lies behind appearances and then it has to test and modify its conclusions, as with optical illusions. All this leads to what we may call subjective reality: the world as it seems to us, based on our past experience.

But science tells us that things are not what they seem. Extending our sensory input by artificial means, using instruments such as microscopes, reveals a very different world. A solid stone is

seen to have an intricate composite structure. Beyond that, modern scientific theories tell us of molecular and atomic structure. The solid stone consists mainly of empty space and the fluctuating waves of quantum mechanics. So which is the “real stone”?

We conclude that there are various levels of ‘reality’

- (a) the human perception of reality
- (b) the scientific description of reality (of increasing complexity as we scale down in size)
- (c) the mathematical form of reality, when everything is described in terms of equations (as in quantum mechanics)

Finally there is the ultimate question. What is reality with human observation removed? For those of a religious disposition there is no problem, as exemplified in the well-known limerick by Monsignor Ronald Knox.

*There once was a man who said God
Must think it exceedingly odd
If he finds that this tree
Continues to be
When there’s no-one around in the Quad[†]*

*Sir, your astonishment is odd
I am always around in the Quad
And that is why this tree
Continues to be
Observed by yours faithfully, God*

There is also an exchange purported to have taken place between Napoleon and Laplace, à propos of ‘La Mécanique Céleste’. Napoleon observed that the book contained no reference to God. Laplace replied “I had no need of this hypothesis”. When Lagrange heard this story his response was “but what a beautiful hypothesis, it explains so much!”

The irony is that the more knowledge we acquire, the further down we dig into the scientific foundations, the more the ultimate mystery deepens.

2. Is knowledge innate or derived from experience?

This was the question examined at length by philosophers such as David Hume and Immanuel Kant. Hume came down firmly on the side of experience. In his view we learn everything through our senses and our interaction with the external world. Kant was more subtle and tried to have it both ways. Eventually he concluded that some knowledge is innate, though most is acquired through experience.

The nature of space, as formalised in Euclidean geometry, was a favourite battle ground. To Kant our understanding of space was innate, while Hume claimed it was learnt by experience. As mathematics and physics progressed, particularly with the discovery of non-Euclidean geometry and later with Einstein’s theory of General Relativity, many scientists assert that Kant has been proved wrong.

In my view this is too shallow an understanding of the issues. It also shows that we need to think more carefully about ‘innate knowledge’ and where it comes from. In Kant’s day few would dispute openly that man was created by God and innate knowledge was part of God’s gift. Nowadays, in the light of Darwinian evolution, we see man as having evolved in the tree of life by a long process of natural selection. Innate knowledge, from this biological perspective, has been ‘learnt’ from

[†] In Oxford a quad is the quadrangular courtyard of a College

experience, not of the individual, but of the human species. In a sense therefore, there is little fundamental difference between the two sides of the philosophical debate.

For an evolutionary biologist there is no contradiction between 'innate knowledge' ignoring non-Euclidean geometry and Einstein. In their struggle for survival our ancestors never encountered 'black holes'. Flat space, as embodied in Euclidean geometry, was all that was needed to escape the clutches of lions and tigers.

Perhaps I can add a personal anecdote on Kant and his theories of space. When I was a student in Cambridge, our mathematical society invited a distinguished professor of philosophy, C D Broad, to give us an evening lecture. He chose to talk on a problem which had much exercised Kant, the difference between right-handed gloves and left-handed gloves. After the lecture, over dinner, I diffidently suggested to Broad that, since Kant's time, we mathematicians had a much better understanding of 'handedness', or chirality as scientists call it. We could even envisage a universe in which a left-handed glove could wander around to distant regions and return to fit your right-hand. Broad would have no truck with this nonsense, who was I a mere student to question the great Immanuel Kant? Suitably chastised I retreated from the battle, but now fifty years later, I still think I was right and philosophy has to respond to advances in our scientific understanding. It is a pity that the term 'Natural Philosophy' has fallen into disuse.

3. What is mathematics?

Mathematics and philosophy have been closely intertwined from the very beginning, their common ground being logic and reason. Natural philosophy, or science as we now call it, arrived from the marriage between the two disciplines. The most fundamental question that faces the mathematical philosopher is:

What is mathematics?

In its most concrete form it can be formulated as:

Are theorems discovered or invented?

According to Plato, mathematics lives in an ideal world, in which dimensionless points, perfect straight lines and circles exist and obey Euclid's laws. What we draw on paper and see in the world around us are approximate imitations of these ideal objects. For a Platonist, mathematics has an existence independent of the real world, its truths or theorems are already in existence just waiting for us mathematicians to stumble on them. This is the world in which theorems are discovered.

All practising mathematicians believe in this platonic view to some degree. As we work to find the truth we sometimes feel as though a door has opened and we see displayed before us what was previously hidden. The beautiful scene was waiting for us to discover.

As an example, consider the celebrated theorem of Pythagoras relating the lengths of the sides of a right-angled triangle: $c^2 = a^2 + b^2$. As a pragmatic fact this was known to the Babylonians who had long tables of such Pythagorean numbers starting with 3, 4, 5 and 5, 12, 13. These were no doubt found experimentally – a vision into the ideal world of the Platonists – although the notion of proof did not emerge till much later with the Greeks. It is hard to dispute that this theorem was a discovery.

There are eminent mathematicians such as Alain Connes and Roger Penrose who are fervid Platonists, for whom the ideal world of mathematics has an enduring existence, independent of humanity. Mathematics, according to them, existed before human beings appeared on the scene and will continue to exist after humanity is extinct. For them mathematics has some of the attributes of God: existence outside time.

But an example of a mathematical idea which, to my mind, represents an invention is $\sqrt{-1}$, the square root of minus one. Since the square of any number (positive or negative) is always positive, there is no number whose square is -1 . However, over the centuries, mathematicians found themselves using the fictional number $\sqrt{-1}$ with great success, so much so that they eventually admitted such 'imaginary' numbers into their world. A good claim can be made that this was the most inventive step taken in the history of mankind. It opened entirely new doors in mathematics and in the 20th century it was found to be essential in the formulation of quantum mechanics.

Familiarity breeds contempt and today's students take $\sqrt{-1}$ in their stride, but the great Gauss said that "the true metaphysics of $\sqrt{-1}$ is not easy".

There are other famous quotations by mathematicians. Kronecker believed that

God created the integers, all else is made by man

and most mathematicians put forward the integers and their properties as prime examples of the ideal world. But, in a *jeu d'esprit* [1], I speculated on what would have happened if evolution had led to higher intelligence emerging not in human beings but in vast jelly-fish that filled oceans. For such beings, which did not meet individual objects, the integers would have no relevance. But real numbers describing things like water pressure, velocity, temperature, would be vital. So one could imagine their mathematics being sophisticated in fluid mechanics but ignorant of number theory. In fact evolution (or God) created man and so the integers. The distinction made by Kronecker evaporates.

Being myself a mathematician I cannot shirk this question of invention versus discovery; what is my view? To be succinct I will simplify and answer by making two statements.

- (1) Mathematics lives in the collective mind of mankind.
- (2) Many theorems exist but we select those we like.

It is hard to dispute (1). It is an empirical statement. A librarian might say that mathematics is contained in all books and articles, but if all libraries suffered the fate of the famous one at Alexandria, mathematical knowledge would survive in the collective human mind. When humanity becomes extinct there is no one left to ask the question, so a strict follower of Wittgenstein would say the question becomes meaningless.

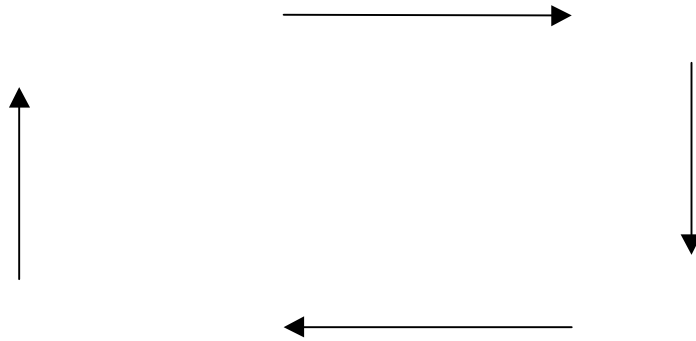
In respect of (2), my view of theorems is that all correct mathematical statements pre-exist our observation of them. In Newton's famous phrase they are like pebbles on the beach and we just pick up one or two because they appeal to us. In other words the raw material is there to be discovered, but we exercise our free will in making a choice – this is where invention enters. Of course this vastly oversimplifies. Invention often entails a major reorganisation; we don't just select pebbles but we put them together to build castles. In principle all such possible castles also exist in advance and we choose which one to build. The beach analogy breaks down at this point, and we have to continue the argument at a more abstract level.

4. What is the relation of mathematics to physics?

There is the famous statement of Galileo

The book of nature is written in the language of mathematics

and it is certainly true that, since his time, mathematics has increasingly become the only way to understand physics. I shall return to this story later. But the relation between mathematics and physics is rather complicated. I can try to summarise it by the following diagram:



The top row encapsulates the use of mathematics to record and organise observations of the natural world, for example, the process by which Kepler took astronomical observations of the planets and deduced the planetary orbits and the laws. The next stage is internal to the world of mathematics and where sophisticated mathematical ideas transform our initial data, for instance Newton's calculus and his laws of motion explained and extended Kepler's observational laws. The new mathematical understanding is then turned into physical theory, as with the inverse-square law of gravitation. This is represented in the diagram by the bottom horizontal arrow. Finally the physical theory is applied back to the real world, as with the discovery of Neptune.

But the relation between mathematics and physics cannot ignore the role of biology and in particular of evolution. Mathematics takes place in the human mind and one can argue that both the content and the format have been conditioned by the nature of the human brain.

Even logic, based on the principle of implication (A implies B), is derived from the causality that we observe in the natural world (A causes B). When our ancestors saw a tiger lurking in the bushes they knew that its next step would be to pounce on them – a fact learnt the hard way! The origin and development of mathematics by mankind has, to a considerable extent, been driven by evolution. In a sense mathematics has been the secret weapon of mankind in its struggle for survival. There is little doubt that, so far, it has been a tremendous success, though we now have to worry that its consequences do not get out of hand and, through one catastrophe or another, lead to our extinction.

5. The human dimension

The biological comments I have just made lead on to a closer examination of science and mathematics as human activities. Not only in the evolutionary struggle for survival but also in the higher realms of intellectual endeavour, it is the human mind that is in charge. It is we who decide what to study, how to organise knowledge and how to erect the great architectural structure that we know as science.

So, what is our driving force? What are the principles that guide us? Where do we get our "master-plan"? Utility and immediate practical need are only modest incentives, they deal

with the short term. They are like the choice of stone that the builder employs. For the grand architectural scheme, the vision in the mind of Michelangelo, we have to seek elsewhere.

Throughout history the aim of science has been for man to understand nature, to acquire the deepest possible insight into its workings and structure. The key here lies in the word 'understand'. What is understanding? It is certainly much more than a mechanical accumulation of facts. Poincaré put this well when he said that science is no more a collection of facts than a house is a collection of bricks.

But whatever understanding is, it is a human attribute. We are not electronic computers that organize and handle vast quantities of data at breath-taking speed. Perhaps a computer may be said to understand a problem but it is very different from human understanding.

Science as we know it is definitely a human enterprise, based on our kind of understanding. It is a cultural activity like art and it is driven by the human search for simplicity and beauty. When we find a simple explanation for a complex phenomenon, such as the rainbow, we claim to have understood it. A simple proof of Pythagoras's theorem enables us to understand all the Babylonian triangles. The inverse square law explains the elliptical planetary orbits.

If simplicity and beauty are the hallmark of understanding, how does the mind actually achieve its objectives in the field of mathematics? On the one hand there is the formal apparatus of logic, proof and computation, the standard tools of the working mathematician. These are like the pencil, paper and laptop of the writer, but what is going on behind the scenes in the mind of the writer or of the mathematician?

Frequently, when asked to describe a piece of mathematics to a lay audience, we avoid technicalities and resort to analogies, as in the use of architecture to indicate structure. We tend to do this apologetically as a poor imitation of the real thing. In fact I believe that analogy is one of the most powerful tools to help achieve understanding. Mathematicians have for instance adopted 'waves' as the term to describe oscillatory behaviour of everything, not just water in the sea. Electro-magnetic waves, quantum wave-functions, seismic waves are familiar examples and sports commentators even talk about the waves of cheers in a football crowd.

Perhaps the most fundamental and widely-used analogy relates to vision, the most complex process taking place in the brain. When a student, confronted by a difficult problem, finally exclaims "I see", vision is being used as a synonym for understanding. To a great extent mental pictures are the key to understanding. This applies very closely to patterns, where a basic unit or cell, gets repeated many times. Such patterns may actually describe visual phenomena, but they can also be abstract patterns, where the cell is a sound, a phrase or just an idea.

The use of analogies, pictures or patterns is fundamental to how we think, both in mathematics and in life. Mathematicians, at all levels, think in these ways and not in the formal language of logic and proof. This is important in teaching: we have to help students to use their imagination not just their computer.

Modern physics

All the questions I have been discussing relating to mathematics, physics and philosophy have become even more relevant in the 20th and now the 21st century. Problems which

were considered archaic dead-ends, about which nothing new could be said have, on the contrary, been brought back to life and are now more relevant than ever. The deeper we dig the more pertinent we find the classical questions, which is why I have chosen my topic today.

I want to review very rapidly the main developments in physics over the past century or so and see where this is leading us. As will become clearer, the role of mathematics has become more and more central to the whole story and this has profound philosophical implications.

For simplicity I list below the main developments in physics, along with the names of the most prominent physicists associated with them. The list is in chronological order, and ends with the uncertain present and future.

Newton	Gravity
Maxwell	Electro-magnetism
Heisenberg	Quantum mechanics
Dirac	Quantum field theory
Witten	String theory
?	?

As we move down the list, following the historical order, we should note two persistent trends. In the first place every step involved a new paradigm, a new concept or point of view, which encountered much initial opposition. Newton's 'action at a distance' without any direct mechanism was found unacceptable by the followers of Descartes. Maxwell's introduction of fields of force in empty space appeared equally revolutionary. Einstein's General Relativity presented great conceptual difficulties, resting as it did on the earlier Special Relativity which had combined space and time. Quantum Mechanics and Quantum Field Theory entered a totally new and bizarre world which Lewis Carroll would have loved to exploit. The most recent era in which string theory attempts to combine Quantum Mechanics with Gravitation moves into totally new territory where space-time has 10 or 11 dimensions (not just the customary four) and strings (one dimensional objects) rather than point-particles are the starting point.

The second historical observation is that, at each step, the theory becomes mathematically more sophisticated. In fact the histories of mathematics and physics, over this whole period, are closely intertwined, even though there have been periods when they seemed to drift about.

The present era, that of strings or their successors, involves mathematics of incredible sophistication, much of it beyond our present understanding. In fact Edward Witten said that string theory was a 21st century idea that was accidentally discovered in the 20th century. In other words, we may need to wait a long while before the full mathematical implications of string theory are properly understood.

Throughout the development of physics which I have been reviewing, there has been a conflict between the philosophy, the physics and the mathematics. Each new theory presented fundamental philosophical problems which were appreciated by their proponents and pounced on by the opposition. The answer of the physicists was always pragmatic: it works. The new theories were fully vindicated by experiments. They were also mathematical triumphs; the equations took charge and in a sense ejected the philosophers.

Not everyone was happy with this outcome. Einstein remained a radical on quantum mechanics, refusing to accept it as an ultimate theory. He had implicit support from Richard Feynman who confessed that “no one really understands quantum mechanics”, though Feynman was himself one of the leaders of the quantum revolution.

It is also interesting to recall that Clerk Maxwell first discovered his famous equations from a mechanistic model, an ‘explanation’ which he subsequently discarded.

I once sat next to the famous Austrian logician and friend of Einstein, Kurt Gödel, who said to me that the trouble with modern physicists is that they no longer aim to ‘explain’, they just ‘describe’. That in a nutshell is the lost battle of the philosophers. Moreover, mathematicians appear as the villains in the play. They have taken the place of the philosophers and equations become the ultimate reality.

The conclusion seems to be that the physical models of the universe, with their history of experimental success, have become totally mathematical. You might think that, as a mathematician, I would welcome this ultimate triumph of my subject, but perversely I am unhappy with the situation and I share Einstein’s misgivings. It is undoubtedly true that the physical models we now have provide incredibly accurate descriptions of most physical phenomena, though the ultimate unification being sought by string theory remains elusive. It is just possible that a new and more refined physical model will be produced which will explain all physical phenomenon and be more Einsteinian in spirit. We should remember that the ultimate goal of science is to understand nature, and while mathematics might be the preferred tool we should also aim at more acceptable philosophical foundations.

At various stages in my lecture I have alluded to God, but these remarks have been made in a slightly humorous way and I did not stress their theological content. But let me now extract from them some tentative remarks of a more serious nature concerning faith. Many physicists like Laplace and Einstein put their faith in mathematics as providing the ultimate explanatory basis for the workings of the universe. Mathematicians like Connes and Penrose, with their Platonic view of mathematics as existing outside space and time, seem to share this faith. On the other hand Lagrange, perhaps tongue in cheek, pointed to God as a “beautiful hypothesis” explaining everything, and Ronald Knox encapsulated it all in a Limerick.

Mathematical physicists believe that there are indeed simple and beautiful mathematical equations that govern the universe, and that the task of the scientist is to search for them. This is an article of faith.

An alternative faith is to believe in a God who created the universe and kindly provided us with laws or equations that we would be able to understand.

There is no conflict between these two faiths, both have their mysteries.

References

[1] Atiyah, M.F. Book Review of CONVERSATIONS ON MIND, MATTER AND MATHEMATICS by Jean-Pierre Changeux and Alain Connes, *Times Higher Educational Supplement*, 29 September 1995

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